Power Law and Fractional Calculus Model of Viscoelasticity

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Mathematical relationships between the power law model and the fractional calculus model of viscoelastic behavior are developed. The (unmodified) power law is shown to be a special case of the general fractional calculus model. The modified power law model is also shown to be closely related to the fractional calculus model. In particular, it is demonstrated that the two models have mathematically similar relaxation spectra. This similarity is seen to produce an asymptotic equivalence of the models at long relaxation times and correspondingly low frequencies of motion in the lower transition and rubbery regions. The divergent behavior of the models at higher frequencies in the transition and glassy regions is presented and, for particular values of the models' parameters, the divergence is seen to be small. Proposals to take advantage of the similarities between power law and fractional calculus models are discussed.

Nomenclature

b_j = viscoelastic compliance parameter
D^α = alpha-order time derivative
E_e = rubber modulus
E_g = glassy modulus
E_0, E_1, E_n = viscoelastic stiffness parameters
$E(\omega)$ = frequency-dependent complex modulus
$H(\tau)$ = relaxation spectrum
i = imaginary coefficient
Im = imaginary part of a complex number
n = parameter of the power law model
Re = real part of a complex number
t = time
$\alpha, \alpha_n, \beta_j$ = fractional orders of differentiation in the fractional calculus model
$\epsilon(t)$ = strain time history
Γ = gamma function
$\sigma(\tau)$ = stress time history
τ_0 = parameter of the power law model
ω = frequency of motion in rad/s

Introduction

HISTORICALLY, the power law representation of viscoelastic behavior has stood apart from the mainstream development of mathematical models of viscoelastic phenomenon in mechanics. The power law evolved as a convenient and accurate tool of the polymer scientist to describe the relaxation behavior while the mechanics community focused its attention on the Maxwell model, the Kevin-Voight model, and complex combinations of these elementary models to describe viscoelastic effects. The mechanics models took the mathematical form of series of time derivatives acting on time-dependent stress and strain fields and predicted an exponential relaxation and creep. This exponential behavior stood in sharp contrast to power law descriptions of relaxation and creep, using fractional powers of time as was first suggested by Nutting.[1]

The process that leads to a reconciliation of these two apparently different approaches begins with the work of Gemant,[2,3] who suggested that fractional-order time differentials be used to describe the frequency-dependent moduli of viscoelastic material. In turn, Scott-Blair[4] and Caputo[5,6] independently suggested that constitutive relationships employing derivatives of fractional order be used as empirical models. The attractiveness of these models was their ability to accurately portray measured properties over decades of frequencies of motion with very few parameters. The weakness of the more classical constitutive relationships is that the macroscopic mechanical behavior of most linear polymers does not exhibit the strong frequency dependence predicted by ordinary derivatives. Consequently, many derivatives (many parameters) are needed in the model alternatively to add and subtract strong frequency dependence, producing an aggregate weaker frequency dependence.

Reconciling these two approaches is to return to the classical mechanics constitutive relationship and replace the ordinary derivatives with fractional-order derivatives acting on the time-dependent stress and strain fields.[7] The resulting predictions of relaxation and creep are expressed in terms of functions that simultaneously exhibit exponential and power law behavior.[8] This link to power law behavior is not just fortuitous: rather, a special case of this fractional derivative constitutive relationship explicitly predicts a power law behavior.

In addition, this special case has a mathematical form virtually identical to the viscoelastic constitutive model based on Rouse's molecular theory[9] for dilute polymer solutions as modified by Ferry, Landel, and Williams[10] for polymer solids with no crosslinking. Thus, the fractional calculus model and the power law model can no longer be viewed as strictly empirical constructs, but instead must be recognized as models consistent with approximations of first principles.[11] The operative restrictions here are that the molecular model is not based on solutions to the atomic bonding equation (Schroedinger's equation) and that its applicability is limited to the rubbery region and the lower frequencies in the transition region of the material.

Nevertheless, the fractional calculus model, viewed as a generalization of the classical viscoelastic constitutive relationship, serves to bring several concepts of viscoelastic modeling into one coherent approach having a foundation in molecular theory. Not only are these fractional calculus models compact analytic descriptors of viscoelastic phenomena, but they are linear operators well suited to use in structural analyses through Laplace transforms. In particular, these constitutive relationships lead to analytic solutions for continuum and dis-

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crete formulations of equations of motion of structures containing both elastic and viscoelastic components.[12,13]

Unfortunately, the power law model does not share in all the strengths of the fractional calculus approach. In particular, the power law predicts as unbounded relaxation modulus for times close to the onset of strain. This unfortunate characteristic has lead to the development of the modified power law, which has a bounded relaxation modulus. This investigation, motivated by a suggestion of Williams,[14] is ultimately intended to bring the modified power law under the umbrella of the fractional calculus approach to viscoelastic modeling. What follows is the development that establishes the mathematical similarities and differences between the modified power model law and the fractional calculus model.

Development

By way of introduction, it is helpful to examine the relationship between the (unmodified) power law approach and the fractional calculus approach to viscoelasticity. In particular, the link between the approaches becomes evident when the stress-strain constitutive relationship[15] for a viscoelastic material is examined:

$$\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t E_{\mathrm{rel}}(\tau)\epsilon(t-\tau)\,\mathrm{d}\tau \tag{1}$$

where the relaxation modulus $E_{\mathrm{rel}}(t)$ is modeled by the simplest form of the power law,

$$E_{\mathrm{rel}}(t) = Et^{-\alpha}, \qquad 0<\alpha<1 \tag{2}$$

This relaxation model yields a constitutive relationship of the form

$$\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{E\epsilon(t-\tau)}{\tau^{\alpha}}\,\mathrm{d}\tau \tag{3}$$

which is functionally equivalent to

$$D^{\alpha}[\epsilon(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\epsilon(t-\tau)}{\tau^{\alpha}}\,\mathrm{d}\tau, \qquad 0<\alpha<1 \tag{4}$$

Equation (4) is the definition of the fractional-order α time derivative[16] of the function $\epsilon(t)$, where $\Gamma(1-\alpha)$ is the gamma function. Hence, the stress is seen to be proportional to the α-order time derivative of the strain history.

$$\sigma(t) = E\Gamma(1-\alpha)\,\mathrm{D}^{\alpha}[\epsilon(t)] \tag{5}$$

This relationship is a special case of the generalized fractional calculus viscoelastic constitutive relationship[7]

$$\sigma(t) + \sum_{j=1}^{J} b_j D^{\beta_j}[\sigma(t)] = E_0\epsilon(t) + \sum_{n=1}^{N} E_n \mathrm{D}^{\alpha_n}[\epsilon(t)] \tag{6}$$

Note that this generalized fractional calculus viscoelastic constitutive relationship is identical to the classical constitutive relationship,[17] except that the traditional derivatives of integer order have been replaced by derivatives of fractional order. Also note that the power law constitutive relationship, derived earlier, is produced from the generalized fractional calculus model by setting $b_j = 0$ for all j and $E_n = 0$ for all n not equal to one. An expanded form of the power law, using more terms to model the relaxation phenomenon,[18]

$$E_{\mathrm{rel}}(t) = E_0 + \sum_{n=1}^{N} \frac{E_n}{\Gamma(1-\alpha_n)}\,t^{-\alpha_n} \tag{7}$$

is seen to lead to an expression for the stress equal to the right side of the generalized fractional calculus constitutive model, Eq. (6).

Unfortunately, the singular nature of the power law's description of relaxation, i.e., infinite modulus at time zero, is inappropriate for the task of accurately portraying the phenomenon, at times close to the onset of the strain. This shortcoming has prompted the use of the modified power law[19] adapted from Eq. (2), with the relaxation modeled by

$$E_{\mathrm{rel}}(t) = E_e + \frac{(E_g - E_e)}{(1 + t/\tau_0)^n} \tag{8}$$

where E_e is the rubbery modulus, E_g is the glassy modulus, and τ_0 and n are parameters chosen to fit the data. There is no restriction that n be an integer. This model predicts a bounded modulus for all non-negative time. Its initial value is the glassy modulus E_g, which then smoothly tapers to the rubbery modulus E_e with increasing time.

Singular relaxation predictions by the fractional calculus model can be suppressed by first insisting that equal numbers of fractional derivatives act on both stress and strain ($J = N$) in Eq. (6) and that the orders of differentiation be equal ($\beta_j = \alpha_n$ when $j = n$). The simplest fractional calculus model embracing these constraints is the case $J = N = 1$ and $\beta_1 = \alpha_1 = \alpha$, and the model takes the form

$$\sigma(t) + b\mathrm{D}^{\alpha}[\sigma(t)] = E_0\epsilon(t) + E_1\mathrm{D}^{\alpha}[\epsilon(t)] \tag{9}$$

The characteristics of this model have been firmly established, as well as the conditions placed on the parameters to insure the model's consistency with the second law of thermodynamics.[8] The relaxation modulus for this model has been shown to be

$$E_{\mathrm{rel}}(t) = E_0 + \frac{\left(\frac{E_1}{b} - E_0\right)}{\pi}\int_0^{\infty} \frac{u^{\alpha-1}\sin\pi\alpha\,\exp(-utb^{1/\alpha})}{(1 + 2u^{\alpha}\cos\pi\alpha + u^{2\alpha})}\,\mathrm{d}u \tag{10}$$

The degree to which this relaxation modulus differs qualitatively and quantitatively from the modified power law's relaxation modulus [Eq. (8)] would clearly gauge the similarities and differences of the two approaches. Unfortunately, a closed-form evaluation of this integral is not possible.

A comparison of the two viscoelastic models is made possible, however, by examining their relaxation spectra. In general, the relaxation spectra $H(\tau)$ is defined as

$$E_{\mathrm{rel}}(t) = E_e + \int_0^{\infty} \frac{H(\tau)e^{-(t/\tau)}}{\tau}\,\mathrm{d}\tau \tag{11}$$

The relaxation spectrum for the modified power law (mpl) has been presented by Williams[19] and takes the form

$$H_{\mathrm{mpl}}(\tau) = \frac{(E_g - E_e)}{\Gamma(n)}\left(\frac{\tau_0}{\tau}\right)^n e^{-\frac{\tau_0}{\tau}} \tag{12}$$

To derive the corresponding relaxation spectrum for the fractional calculus model, one need only perform a change in variable of $ub^{-1/\alpha} \Rightarrow \tau^{-1}$ in the expression for its relaxation curves [Eq. (10)] and note that[20]

$$\frac{\sin\pi\alpha}{\pi} = \left(\Gamma(\alpha)\Gamma(1-\alpha)\right)^{-1} \tag{13}$$

The resulting relaxation spectrum for the fractional calculus (fc) model is

$$H_{\mathrm{fc}}(\tau) = \frac{\left(\frac{E_1}{b} - E_0\right)}{\Gamma(\alpha)}\frac{\left[\frac{b}{\Gamma(1-\alpha)}\right]}{\tau^{\alpha}}\left[1 + 2\left(\frac{b}{\tau^{\alpha}}\right)\cos\pi\alpha + \left(\frac{b}{\tau^{\alpha}}\right)^2\right]^{-1} \tag{14}$$

Notice that both spectra have very similar mathematical structure. The first term in both is the glassy modulus minus the

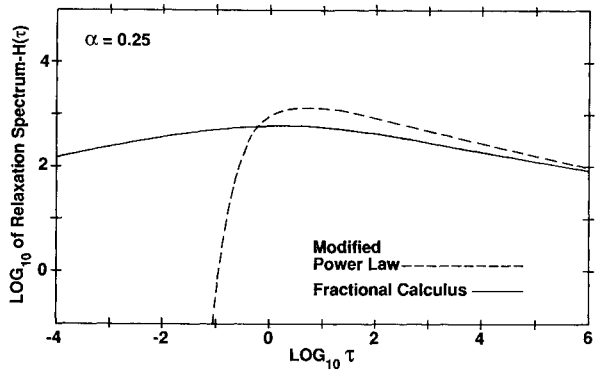

Fig. 1 Comparison of the relaxation spectra for $\alpha = 0.25$.

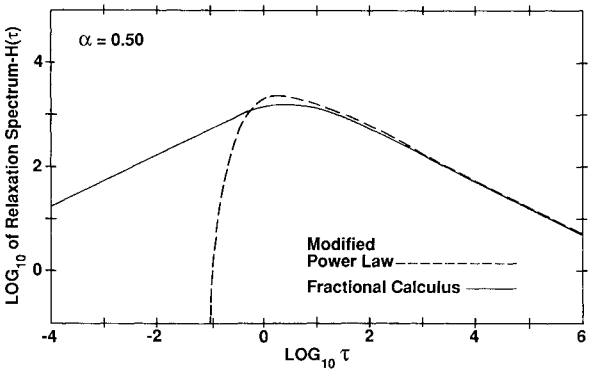

Fig. 2 Comparison of the relaxation spectra for $\alpha = 0.50$.

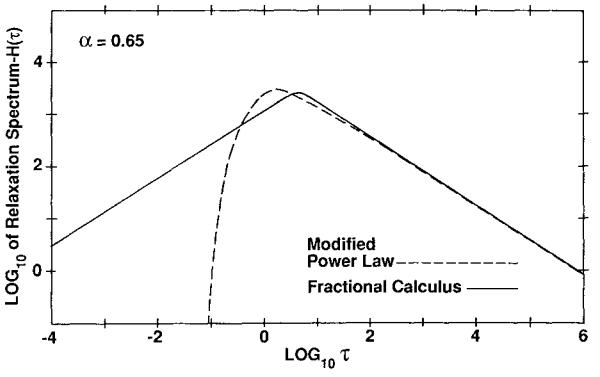

Fig. 3 Comparison of the relaxation spectra for $\alpha = 0.65$.

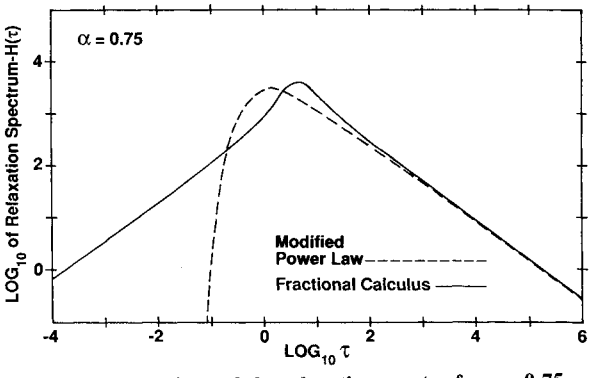

Fig. 4 Comparison of the relaxation spectra for $\alpha = 0.75$.

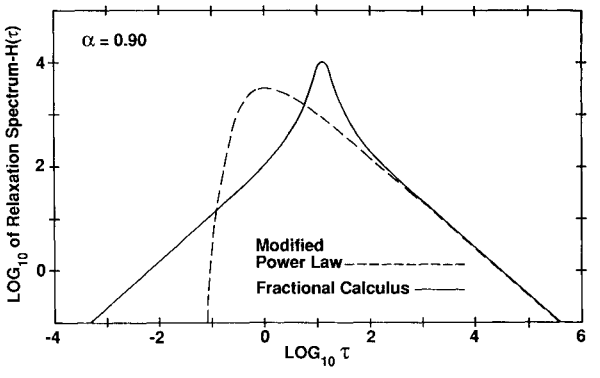

Fig. 5 Comparison of the relaxation spectra for $\alpha = 0.90$.

rubbery modulus divided by a gamma function. The argument of the gamma function is in both cases the order of the singular function of τ appearing in the second term of each of the spectra.

The third term in both spectra are functions having an asymptote of one for $\tau \to \infty$ and having value of zero for $\tau = 0$. For the modified power law, this function tends to zero for small τ as the inverse of an exponential function of inverse argument. For the fractional calculus model, this function tends to zero as $\tau^{2\alpha}$. In both models, the third terms dominate the singular second terms for small τ, driving the spectra to zero for $\tau = 0$.

To complete the comparison between the spectra and their description of the relaxation phenomenon, it is necessary to relate the parameters of one model to the parameters of the other. After the requirement is met that both models produce the same values for the rubbery and glassy moduli of any specific material, the resulting equalities relating fractional calculus model parameters to modified power law parameter are

$$E_0 = E_e \tag{15}$$

and

$$\frac{E_1}{b} = E_g \tag{16}$$

Furthermore, one would prefer to match the relaxation spectra of the models where the most credible relaxation data are available, that is, for large values of τ. Asymptotically matching the two relaxation spectra *be* for large τ yields two more equalities relating the parameter of the models:

$$\alpha = n \tag{17}$$

$$\frac{b}{\Gamma(1-\alpha)} = \tau_0^n \tag{18}$$

Because both models have four parameters, these four relationships are sufficient to transform one model into the other. Given E_0 and α, E_e and n can be determined, or conversely. With α and n known, b or τ_0 may be determined from the other. Finally, with b known, knowledge of E_1 or E_g is sufficient to determine the other. In addition, these relationships produce complete equivalence in the first two terms in the expressions for the relaxation spectra, as given by Eqs. (12) and (14). The difference between the two spectra is now completely confined to the difference between the third terms.

To compare the spectra, the parameters are given the values $E_e = 10$, $E_g = 10^4$, $\tau_0 = 1$; and α is given the values shown in Figs. 1-5. The major differences between the spectra occur at values of $\tau < \tau_0$. These differences are directly attributable to the comparatively rapid decrease of the exponential term in the modified power law spectra as τ becomes much less than τ_0. Conversely, the fractional calculus spectra appear to be more robust in describing materials where short relaxation times, $\tau \ll \tau_0$, are not insignificant contributors to the response of the material. Note that, for $\tau > \tau_0$, there is somewhat better agreement in the spectra and, for $\tau \gg \tau_0$, the spectra are asymptotically equivalent.

To demonstrate the similarities in the frequency-dependent moduli predicted by the models, we begin by substituting the expression for the relaxation function of the modified power law, Eq. (8), into the general uniaxial viscoelastic constitutive relationship, Eq. (1), producing

$$\sigma(t) = E_e\epsilon(t) + (E_g - E_e)\tau_0^n \frac{d}{dt}\int_0^t \frac{\epsilon(t-\tau)}{(\tau+\tau_0)^n}\,d\tau \tag{19}$$

Performing a change of variables, $\tau+\tau_0 \rightarrow \xi$, and separating the resulting integral into two parts yield

$$\sigma(t) = E_e\epsilon(t) + (E_g - E_e)\tau_0^n \left(\frac{d}{dt}\int_0^{t+\tau_0} \frac{\epsilon(t+\tau_0-\xi)}{\xi^n}\,d\xi - \frac{d}{dt}\int_0^{\tau_0} \frac{\epsilon(t+\tau_0-\xi)}{\xi^n}\,d\xi\right) \tag{20}$$

Assuming the strain to be sinusoidal ($\epsilon(t) = \epsilon_0 e^{i\omega t}$), taking time large to allow transients to decay, and dividing the resulting expression for the steady-state stress by the amplitude of strain produce the frequency-dependent complex modulus for the modified power law.

$$E(\omega) = E_e + (E_g - E_e)\tau_0^n\left(\Gamma(1-n)(i\omega)^n - (i\omega)^n\int_0^{i\omega\tau_0} \frac{e^{-x}}{x^n}\,dx\right)e^{i\omega\tau_0} \tag{21}$$

For comparison, the frequency-dependent complex modulus for the four-parameter fractional calculus model

$$E(\omega) = \frac{E_0 + E_1(i\omega)^\alpha}{1 + b(i\omega)^\alpha} \tag{22}$$

is obtained by taking the Fourier transform of Eq. (9) and noting the special property of the transform of a fractional-order derivative,

$$F[D^\alpha[x(t)]] = (i\omega)^\alpha F[x(t)] \tag{23}$$

Approximating the denominator of Eq. (22) for low frequencies,

$$\frac{1}{1 + b(i\omega)^\alpha} \simeq 1 - b(i\omega)^\alpha \qquad \text{for } b\omega^\alpha \ll 1 \tag{24}$$

and retaining only those terms of order $(i\omega)^\alpha$ and lower in the modulus yield this low-frequency approximation.

$$E(\omega) \simeq E_0 + (E_1 - E_0 b)(i\omega)^\alpha \qquad b\omega^\alpha \ll 1 \tag{25}$$

The corresponding low-frequency approximation of the power law modulus [Eq. (21)] is

$$E(\omega) \simeq E_e + (E_g - E_e)\tau_0^n\Gamma(1-n)(i\omega)^n, \qquad \omega\tau_0 \ll 1 \tag{26}$$

The integral term in the original expression is negligible because its domain of integration vanishes as $\omega\tau_0 \rightarrow 0$, and the singularity is weak for $n < 1$. Notice that employing the transformation of parameters, Eqs. (15-18), to either of these low-frequency approximations produces the other. In other words, the transformation of parameters motivated by matching relaxation spectra for large τ produces frequency-dependent moduli that also match at low frequency, which should come as no surprise.

At high frequencies, $b\omega^\alpha \gg 1$ and $(\omega\tau_0)^n \gg 1$, the real parts of both moduli are again equal. This result stems from the constraint imposed earlier that both models have the same glassy modulus. In the case of the fractional calculus modulus, Eq. (22), this is seen to be E_1/b. To examine the high-frequency behavior of the modified power law's modulus, Eq. (21), we need to approximate the integral, an incomplete gamma function, with an asymptotic expansion for large $i\omega\tau_0$. The resulting first term of the expression for the frequency-dependent modulus is

$$E(\omega) \simeq E_g \tag{27}$$

as expected.

Having assured the equivalence of the two models at frequencies in the rubbery and lower transition regions, it is instructive to compare their frequency-dependent moduli at higher frequencies in the transition and glassy regions. For the power law, the real and imaginary parts of the power law moduli are computed using

$$\text{Re}\,[E(\omega)] = E_e + \int_0^\infty \frac{H(\tau)(\omega\tau)^2}{1+(\omega\tau)^2}\frac{d\tau}{\tau} \tag{28}$$

and

$$\text{Im}\,[E(\omega)] = \int_0^\infty \frac{H(\tau)(\omega\tau)}{1+(\omega\tau)^2}\frac{d\tau}{\tau} \tag{29}$$

respectively, and are to be compared to the predictions of Eq. (22) for the fractional calculus model. Comparisons of the real and imaginary parts of the moduli for five selected values of α are given in Figs. 6-15. These moduli are calculated using the same parameters as before and are plotted as functions of frequency ω in rad/s.

For the values of α selected in Figs. 6-15, varying degrees of similarity between the modified power law and fractional calculus moduli are observed. The degree of similarity between the moduli roughly corresponds to the degree of similarity observed earlier in the spectra (Figs. 1-5). There is reasonably good agreement between the real parts of the moduli (storage moduli), but substantial differences appear in the imaginary parts (dissipation moduli). The best overall agreement occurs

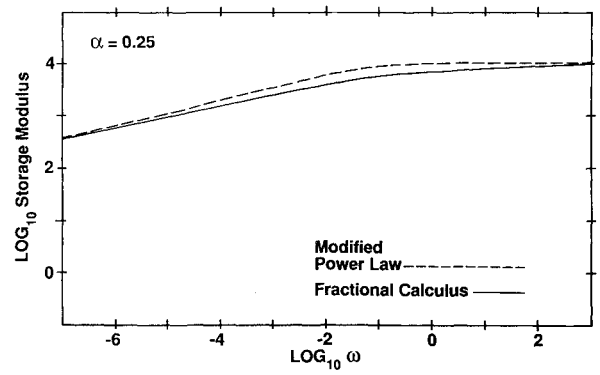

Fig. 6 Comparison of the storage moduli for $\alpha = 0.25$.

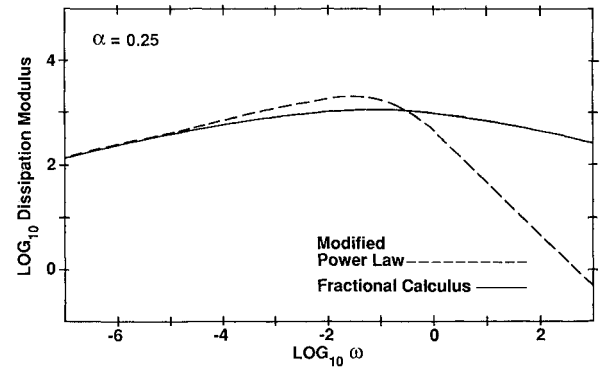

Fig. 7 Comparison of the dissipation moduli for $\alpha = 0.25$.

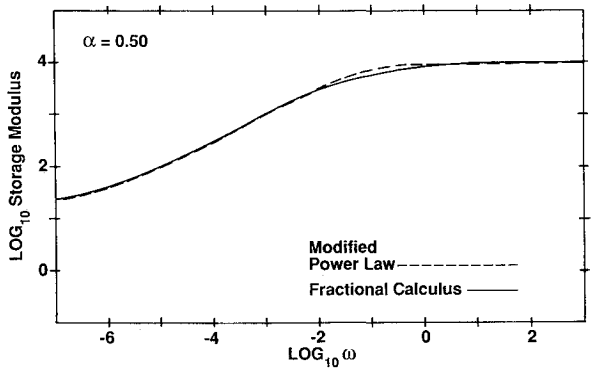

Fig. 8 Comparison of the storage moduli for $\alpha = 0.50$.

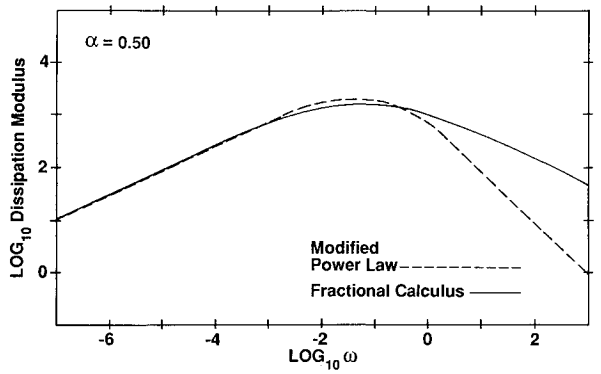

Fig. 9 Comparison of the dissipation moduli for $\alpha = 0.50$.

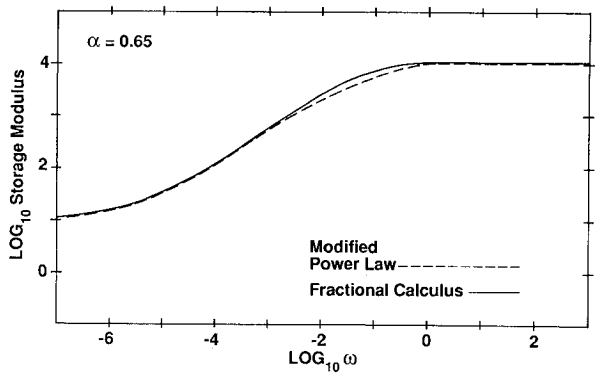

Fig. 10 Comparison of the storage moduli for $\alpha = 0.65$.

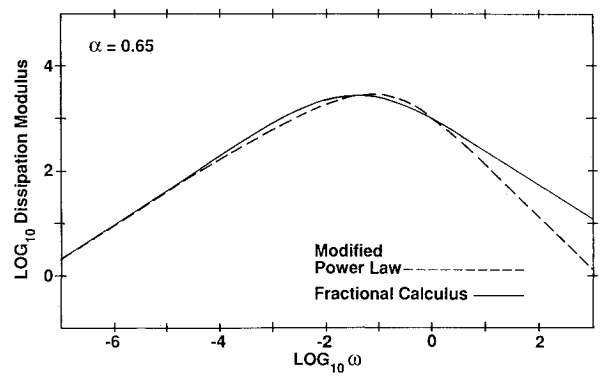

Fig. 11 Comparison of the dissipation moduli for $\alpha = 0.65$.

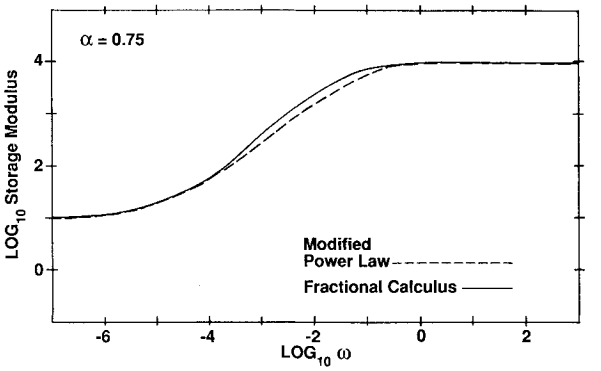

Fig. 12 Comparison of the storage moduli for $\alpha = 0.75$.

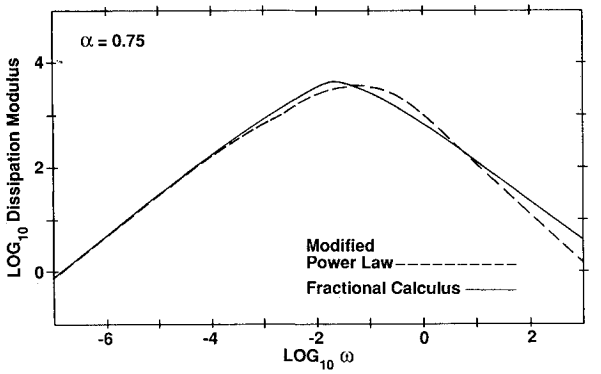

Fig. 13 Comparison of the dissipation moduli for $\alpha = 0.75$.

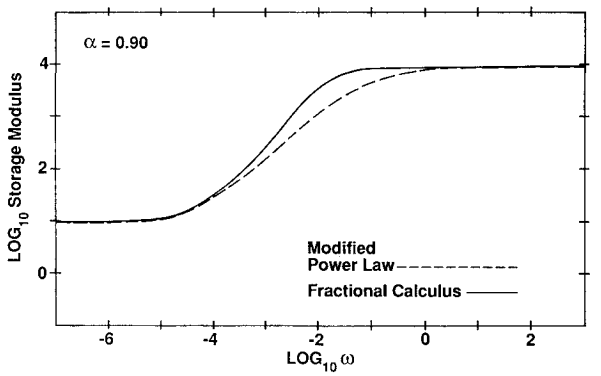

Fig. 14 Comparison of the storage moduli for $\alpha = 0.90$.

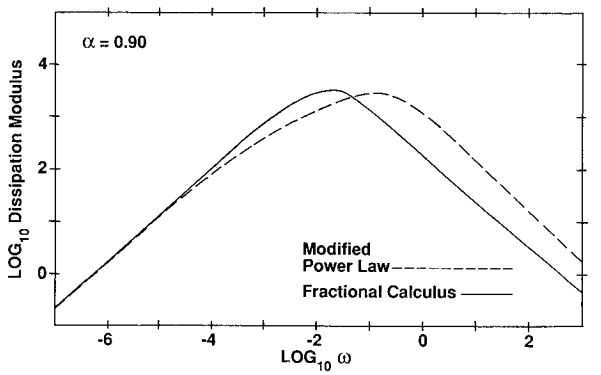

Fig. 15 Comparison of the dissipation moduli for $\alpha = 0.90$.

for $\alpha = 0.65$, whereas $\alpha = 0.25$ and $\alpha = 0.90$ produce the largest disparities. In all case, the asymptotic equivalence of the models is seen to hold for $\omega\tau_0 < 10^{-4}$. Recall that this asymptotic equivalence resulted from relationships between the parameters of the two models.

We can employ these relationships to establish the degree of agreement between viscoelastic material properties obtained under time domain testing with those obtained under frequency domain testing. We could make experimental measurements of the material's frequency-dependent modulus, curve-

fit to determine the parameters in the fractional calculus model, transform these parameters into the parameters of the modified power law, and use the resulting time-dependent stress relaxation curve to compare with the results of stress relaxation tests. Obviously, we can reverse this process by starting with time domain measurements and working backward to a comparison in the frequency domain. In either case, the comparison of the results of two essentially different tests to determine material properties would greatly increase the confidence with which we can apply either model in describing viscoelastic phenomena.

There is also a very practical application of these results. Notice that the expression for the modified power law's stress relaxation modulus has simple mathematical form, whereas its frequency-dependent modulus is anything but simple. Conversely, the fractional calculus model's frequency-dependent modulus has very simple mathematical form whereas the expression for its stress relaxation modulus is cumbersome to evaluate. In those cases in which the relaxation spectra and frequency-dependent moduli predicted by both models are essentially equivalent, we could use Eq. (8) to approximate the stress relaxation modulus for the fractional calculus model and Eq. (22) to approximate the frequency-dependent modulus of the modified power law model after using Eqs. (15–18) to transform the constants. Thus, advantage may be taken of those cases in which the models exhibit near equivalence to simplify analyses.

Closure

The modified power law and the fractional calculus model are different but closely related models of viscoelastic behavior. The similar mathematical form of their respective relaxation spectra spawn relationships between the sets of parameters for both models. These relationships establish equivalent frequency-dependent behavior of the models in the rubbery region, the lower transition region and, for many practical applications when lossy behavior is unimportant, the glassy region as well. The difference in the models lies in their behaviors at the upper frequencies of the transition region or, equivalently, at short relaxation times.

In spite of these differences, it appears that, in some cases, the modified power law and the fractional calculus model can, with care, produce very similar material characterizations. If, however, we are interested in using the model in structural analyses in which predictions of dynamic response are sought, the fractional calculus model is clearly superior. In particular, the fractional calculus approach employs linear operators leading to the fractional-order differential equations of motion that have analytic, causal solutions for continuum and discrete formulations with general loading conditions. The results of this work show that the characterization of a material in terms of the parameters of an inverse power law need not impede the analysis of such a material in a vibratory environment. Rather, the parameters may be transformed into those of the fractional calculus model, and the benefits of that model retained. Thus, although the modified power law does not share all of the advantageous characteristics with the fractional calculus model, the strong similarities between the models can be used to facilitate the process of modeling viscoelastic phenomena with greater care and confidence.

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